Symmetry in Boolean Constraints: Some Complexity Issues

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1 Introduction

The recognition and use of symmetry within automated reasoning, explicitly or implicitly, is a powerful technique. For example, the well-known n-queens problem can be tackled by noting that the eight geometric symmetries of the problem translate into an invariance group of the set of clauses; this reduces the search space, as was noted already by [Gla74]. More recently, various proof systems have been proposed incorporating symmetry rules [Kri85, BdlT96]. Larger symmetry groups can be obtained by looking for logical equivalence of formulae under appropriate transformations: the 6-queens problem, for example has just four distinct models viewed in this way. The larger the known symmetry group of the problem, the fewer distinct points there are to visit in the search space and the greater will be the benefits of the symmetry knowledge. On the other hand, computing with larger groups also adds a cost factor, but this has usually been found to be a good trade-off as the result of the polynomial complexity of many relevant permutation group algorithms.

This raises the question of how difficult it is to compute the group of logical, or semantic, symmetries of a propositional formula. Boy de la Tour showed that his problem GSYMM of computing a set of generators for the group of semantic symmetries is at least as hard as the satisfiability problem itself in the sense that an efficient computation of the semantic symmetries would imply an efficient algorithm for the satisfiability problem. This negative result would seem to indicate that there is little point expending effort to compute all the symmetries in order to improve the efficiency of checking satisfiability of a problem. However, even partial knowledge of the symmetry of a problem can give computational gains.

In the special case of boolean constraints, we establish some complexity results connecting group actions and propositional formulae.

2 Groups acting on boolean formulas

We start by defining two groups over the set of literals as follows, where Ω is the set of boolean atoms.

$$\Delta \quad \equiv_{def} \quad \langle (x\ \neg x) \mid x \in \Omega \rangle$$

Δ is thus the group generated by all independent 'flips' between x and $\neg x$; we define $S(\Omega)^{\neg}$ to be the symmetric group on Ω with a naturally induced parallel action on negative literals. Finally the group W combines the previous groups: $W \equiv_{def} \Delta S(\Omega)^{\neg}$.

We wish to define an action of W on the boolean terms Wff. We extend the action on literals in the

[1] The results described are due to James Molony.

"obvious" way by requiring e.g. $(p \wedge q)^g = p^g \wedge q^g$. We believe this is the "right" base group for the study of symmetry of Boolean constraints (for example, the actions preserve logical validity).

We operate with finite subgroups $W(\vec{x})$ for sequences $\vec{x} \subset \Omega$; $W(\vec{x})$ has size $2^n\ n!$.

3 Results

It is known from [FHL80] that, surprisingly perhaps, that given a set of generators for $G \leq S(\vec{x})$, one can determine in polynomial time the order of G and whether a given permutation g is in G.

We show various complexity results. For example, verifying a conjecture in [BdlT96], we show that the following problem is co-NP-complete:

> Given a wff $\phi(\vec{x})$ and a permutation $g \in W(\vec{x})$, is g is a member of the semantic symmetry group of ϕ.
>
> Is g a semantic symmetry of ϕ? Formally is it the case that $\models \phi^g \leftrightarrow \phi$ is true?

We also show that the following problem is Σ_2^p-complete:

> Given wff $\phi(\vec{x})$ and group $G \leq W(\vec{x})$, is there a satisfiable $\psi(\vec{x})$ such that $\psi \models \phi$ and ψ is semantically invariant under G?
>
> Formally, is it the case that $\bigwedge_{g \in G} \phi^g$ is satisfiable?

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