Spatial Reasoning with Intuitionistic Logic

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The formalisation of the spatial knowledge centred around the notion of *region*, as it is normally included in real-world reasoning, is a theoretical field (Qualitative Spatial Reasoning, QSR) which can be relevant for applications in spatial information systems [CBGG97].

Overlapping, ol, and *connection*, rc, together with their complements (resp. *disjointness*, dj, and *disconnection*, dc), are relations between regions which are particularly significant in QSR. Intuitively, connection holds between two regions, whenever there is not a third one which separates them completely one from the other, whereas overlapping holds when the two regions have a sub-region in common; in topology, this distinction can be drawn in terms of open and closed sets. Connectedness (also *self-connectedness*, sc) is a topological property of spaces and of their subsets, representing intuitively the idea that something is "all of one piece".

The topological distinction between closed and open sets can be problematic from the point of view of real-world reasoning. This has led to introduce a region-based theory of spatial relations (Region Connection Calculus, *RCC* [RCC92]), where the distinction between connection and overlapping is made, whereas the one between open and closed sets is withheld. [Ben96] and [SW97] have shown that the algebraic structure of *RCC* bears quite close a relationship to Heyting algebras.

In this abstract we are considering an approach to QSR which is essentially based on intuitionistic propositional second-order logic (*IPL2*).

Intuitionitic predicate calculus has a well-known topological interpretation [RS63]; atomic formulae are mapped to open sets; logical operators are associated to Heyting algebra operations that can be defined on any topology; this gives the basics for a spatial interpretation of the formulae. Constructiveness (i.e. disjunction property and existence property together) has a topological counter-part in strong compactness; this suggests a restriction to strongly compact spaces. Second-order propositional logic can be regarded as a predicate calculus, where the domain is constant and equal to the universe of interpretation. Intuitionistic predicate calculus with constant domain is complete with regard to T_0, Alexandroff spaces [Gab81] (Alexandroff topologies are closed with regards to arbitrary intersection). This immediately suggests that *IPL2* is complete with regard to strongly compact, T_0, Alexandroff spaces (*IPL2 spaces*).

Fourman and Scott ([FS79] section 1.4) propose a characterisation for connected space in a topological model, which, differently from others [FG82], does not require the preliminary introduction of a notion of consistency (related to *non-emptiness*, ne). Their meta-theoretic definition suggests the possibility of a theoretic *IPL2* definition of connectedness for open sets in an *IPL2* space, relying on a spatial interpretation of $\rightarrow$ as *being part of*, and of $\wedge, \vee$ as different forms of overlay.

Both the spatial interpretation and the semantical one can be put at work in *IPL2*. This allows us, without adding extra axioms, to define relations quite close, from the point of view of the logical properties, to the corresponding ones in *RCC*, relying essentially on the difference between $\mathsf{sc}(\alpha \vee \beta)$ and $\mathsf{sc}(\sim\sim(\alpha \vee \beta))$; this has been verified by the hand-proving of some *IPL2* theorems. Future work should involve mechanisation of the proofs using an interactive theorem-prover.

Regions here are represented by non-empty open sets. Non-emptiness can be defined using sc. dj and ol can be expressed respectively by $\sim(\alpha \wedge \beta)$ and $\mathsf{ne}(\alpha \wedge \beta)$. In the "positive" definitions (1, 2, 3) strong compactness is required in order to preserve the general spatial meaning.

1 $\mathsf{sc}(\alpha) \ =_{def}\ \forall xy.\ (\alpha \to x \vee y) \wedge \sim(x \wedge y) \to (\alpha \to x) \vee (\alpha \to y)$

2 $\mathsf{ne}(\alpha) \ =_{def}\ \forall xy.\ \mathsf{sc}(x) \wedge \mathsf{sc}(y) \wedge (\alpha \to x \wedge y) \to \mathsf{sc}(x \vee y)$

3 $\mathsf{dc}(\alpha, \beta) \ =_{def}\ \forall xy.\ (x \to \alpha)\ \wedge\ (y \to \beta)\ \wedge\ \mathsf{sc}(\sim\sim(x \vee y))\ \to\ \sim x \vee \sim y$

4 $\mathsf{rc}(\alpha, \beta) \ =_{def}\ \exists xy.\ \mathsf{ne}(x)\ \wedge\ \mathsf{ne}(y)\ \wedge\ (x \to \alpha)\ \wedge\ (y \to \beta)\ \wedge\ \mathsf{sc}(\sim\sim(x \vee y))$

We end with some points that might be interesting for further study, in addition to the proof-checking work:

IPL2 spaces, unless trivial ones, cannot be Hausdorff; however they can be finite. Possible relationship with granular representation in discrete geometry should be investigated.

Although *IPL2* is undecidable, it is possible to represent it in *IPL* [Pit92]; of course the representation is not complete; it would be interesting to know how this translation affects the spatial interpretation.

References

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