On the relationship between decidable fragments, non-classical logics, and description logics

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Abstract. The guarded fragment and its extensions and subfragments have often been considered as a framework for investigating the properties of description logics. But there are other decidable fragments which all have in common that they generalise the standard translation of $\mathcal{ALC}$ to first-order logic. We provide a short survey of some of these fragments and motivate why they are interesting with respect to description logics.

In this abstract we focus on the description logic $\mathcal{ALC}$ and the extensions of $\mathcal{ALC}$ by disjunction, conjunction and negation on roles and discuss their relationship to less well known logics including Boolean modal logic, the two-variable fragment, the dual of the Maslov's class K, Quine's fluted logic, and the positive restrictive quantification fragment PRQ. We compare the expressive power of the formalisms, and discuss the applicability of various inference methods established in the field of non-classical logics to description logics.

We denote the extension of $\mathcal{ALC}$ by the role-forming operators $r_1, \ldots, r_n$ by $\mathcal{ALC}(r_1, \ldots, r_n)$. The definition of the standard semantics of $\mathcal{ALC}$ [8] and its extensions by the role operators disjunction, conjunction, negation and inverse, indicates that these languages can be considered as subfragments of first-order logic. One of these fragments is the guarded fragment [1]. However, one important property of the guarded fragment is that the guards, that is the atoms we obtain from the translation of roles, are always positive. Therefore, $\mathcal{ALC}(\neg)$ and its extensions contain concepts whose translation will not result in guarded formulae. An example of an $\mathcal{ALC}(\sqcap, \neg)$ concept which cannot be translated into a guarded formula is the set of cheese lovers, that is $\forall\neg(\mathit{likes} \sqcap \mathit{eats}).\neg\mathit{Cheese}$.

Making economical use of first-order variables concepts of $\mathcal{ALC}(\neg)$ and many other description logics can be embedded into finite-variable function-free fragments of first-order logic. One of them is the *two variable fragment* FO^2 which consists of those formulae of first-order logic that can be written using only two variables. The description logic which has the same expressive power as the two variable fragment is $\mathcal{ALB}$ [4], i.e. $\mathcal{ALC}$ extended with full Boolean operations on roles and the inverse operator. An extension of FO^2 is the dual of Maslov's class K, denoted by $\overline{\mathrm{K}}$ [5]. Consider the following formula φ_1 which defines the concept mwc as a subset of married couples with a child: $\forall x\forall y(mwc(x,y) \rightarrow (married(x,y) \wedge \exists z(has\_child(x,z) \wedge has\_child(y,z))))$. The formula φ_1 is not in the guarded fragment and also not in FO^2, but it is in $\overline{\mathrm{K}}$. In contrast, the formula $\forall x\forall y(mwd(x,y) \rightarrow \forall z(have\_child(y,x,z) \rightarrow doctor(z)))$ which describes the concept mwd as a subset of married couples with a child who is a doctor, is guarded, but not in $\overline{\mathrm{K}}$. So, the guarded fragment is not a subfragment of $\overline{\mathrm{K}}$ nor is $\overline{\mathrm{K}}$ a subfragment of the guarded fragment. However, $\overline{\mathrm{K}}$ contains a variety of classical solvable fragments, namely the monadic class, the initially extended Skolem class, the Gödel class, and FO^2 as well as a range of non-classical logics, like a number of extended modal logics, many description logics, reducts of representable relational algebras [3].

The concept $mwmc$ of married couples whose all children are married, defined as follows: $\forall x_1\forall x_2(mwmc(x_1,x_2) \leftrightarrow (married(x_1,x_2) \wedge \forall x_3(have\_child(x_1,x_2,x_3) \rightarrow \exists x_4 married(x_3,x_4))))$ is not guarded but also not in $\overline{\mathrm{K}}$. It belongs to yet another solvable fragment of first-order logic, namely *fluted logic*. In fluted logic the relational atoms may be negated, and consequently $\mathcal{ALC}(\neg)$, $\mathcal{ALC}(\sqcup, \sqcap, \neg)$, and the Boolean modal logic which cannot be embedded into the guarded fragment can be embedded into fluted logic [7]. Moreover fluted logic can be extended with converse on relations while still preserving decidability [6]. Fluted logic with converse allows for the embedding of standard modal logics K, KT, KD, KB, KTB, more expressive logics, like $\mathcal{ALB}$ and the

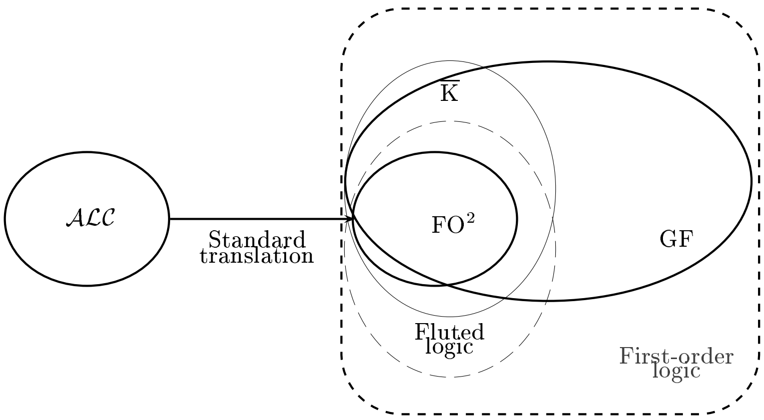

Fig. 1. The relationship of $\mathcal{ALC}$, FO^2, GF, $\overline{\mathrm{K}}$, and fluted logic

corresponding modal logic $K_{(m)}(\cap, \cup, \neg, {}^{\smile})$, and also FO^2. The relationship among the decidable fragments, discussed so far, in this abstract is summarised in Figure 1.

However, if instead of extensions of $\mathcal{ALC}$ by role-forming operators we look at products of $\mathcal{ALC}$ with modal logics, for example, basic modal logic, then there are examples which fall in neither of the solvable fragments of first-order logic we have looked at so far. Consider the sentence: *Minis are what Don believes to be a slow car*. The sentence is expressible in $K_{\mathcal{ALC}}$ as follows. $Minis \subseteq [Don]_{believes}$. $slow\_car$ Its standard translation to first-order logic is the formula φ_3.

$$\forall x(Minis(reality, x) \rightarrow \forall w(believes\_Don(reality, w) \rightarrow slow\_car(w, x)))$$

This formula is neither guarded nor fluted and is also not in $\overline{\mathrm{K}}$. However, it belongs to the so-called positive restrictive quantification fragment (PRQ) introduced in [2]. Even though PRQ is not solvable for fragments of PRQ that have the finite model property, there is a decision procedure in the form of an extended positive tableaux method [9]. The method does not only detect unsatisfiability but also generates finite models if they exist. Since $\mathcal{ALC}$ and many of its extensions have the finite model property, this procedure provides a general, sound, complete, and terminating method for solving the satisfiability problem for these logic without the necessity of additional correctness or termination proofs. The procedure is also applicable to products of modal and description logics which have the finite modal property such as $K_{\mathcal{ALC}}$ [10]. The product of $\mathcal{ALC}$ with basic modal logic K has the finite model property, therefore the considered example in $K_{\mathcal{ALC}}$ is covered by the procedure.

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