A hypersequent calculus for Abelian Logic

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1 Introduction

Abelian logic **A** was introduced independently by Meyer and Slaney in [2] and Casari in [3] as the logic of *lattice-ordered abelian groups*, motivated by the former as a logic of *relevance* and by the latter as a logic of *comparison*. Recently, extending a result in [2], it has been shown that the so-called material fragment of **A** coincides with Łukasiewicz's infinite-valued logic **Ł** [5]. Here we present a cut-free hypersequent calculus for **A**, employing rules familiar from Avron's calculi for Gödel-Dummett logic **LC** and the relevance logic **RM** [1]. Since **Ł** is a fragment of **A** we are also able to fill a hole in the literature identified in [4] and give a hypersequent calculus for **Ł**.

2 Abelian Logic

Definition 1 (Abelian Logic, A) $\to, +, \wedge, \vee$ *and* t *are primitive.* **A** *has the following definitions, axioms and rules:*

$$\begin{array}{ll} D\leftrightarrow & A \leftrightarrow B = (A \to B) \wedge (B \to A) \\ D\neg & \neg A = A \to t \end{array}$$

$$\begin{array}{ll}
A1 & A \leftrightarrow (t \to A) \\
A2 & ((A \vee B) \to C) \leftrightarrow ((A \to C) \wedge (B \to C)) \\
A3 & ((A + B) \to C) \leftrightarrow (A \to (B \to C)) \\
A4 & A \to ((A \to B) \to B) \\
A5 & (A \to B) \to ((B \to C) \to (A \to C)) \\
A6 & A \wedge B \to A \\
A7 & A \wedge B \to B \\
A8 & ((A \to B) \wedge (A \to C)) \to (A \to (B \wedge C)) \\
A9 & (A \wedge (B \vee C)) \to ((A \wedge B) \vee (A \wedge C)) \\
A10 & ((A \to B) \to B) \to A
\end{array}$$

$$(mp)\quad \frac{A \to B, A}{B} \qquad (\wedge I)\quad \frac{A, B}{A \wedge B}$$

Definition 2 (Lattice-Ordered Abelian Group) *A lattice-ordered abelian group (abelian l-group) is an algebra* $\langle G, +, \vee, \neg, t\rangle$ *with binary operations* $+$ *and* $\vee$*, a unary operation* $\neg$ *and a constant* t*, satisfying the following equations:*

$$\begin{array}{ll}
a1 & t + a = a \\
a2 & a + b = b + a \\
a3 & (a + b) + c = a + (b + c) \\
a4 & a + \neg a = t \\
a5 & a \vee b = b \vee a \\
a6 & (a \vee b) \vee c = a \vee (b \vee c) \\
a7 & a = a \vee a \\
a8 & a + (b \vee c) = (a + b) \vee (a + c)
\end{array}$$

We also define $a \wedge b = \neg(\neg a \vee \neg b)$, $a \to b = \neg(a + \neg b)$ *and* $a \leftrightarrow b = (a \to b) \wedge (b \to a)$

Theorem 1 (Characterisation theorem for A) *The following are equivalent: (1)* ϕ *is a theorem of* **A**. *(2)* ϕ *is valid in all abelian l-groups. (3)* ϕ *is valid in* $(\mathbb{Q}, +, \vee, -, 0)$.

3 A hypersequent calculus

Definition 3 (Hypersequent) *A hypersequent is a multiset of sequents, written:* $\Gamma_1 \vdash \Delta_1 | \ldots | \Gamma_n \vdash \Delta_n$ *where* Γ_i *and* Δ_i *are finite multisets of formulae for* $i = 1, \ldots, n$.

Definition 4 (Interpretation) *The interpretation of a sequent* $S = A_1, \ldots, A_n \vdash B_1, \ldots, B_m$ *is* $\phi_S =$

$(A_1 + \ldots + A_n) \to (B_1 + \ldots B_m)$. *The interpretation of a hypersequent* $H = S_1 | \ldots | S_n$ *is* $\phi^H = \phi^{S_1} \vee \ldots \vee \phi^{S_n}$.

Definition 5 (LA: A hypersequent calculus for A) ***LA*** *has the following rules:*

Axioms

$$(id) \quad A \vdash A \qquad (emp) \quad \vdash$$

Structural rules

$$(EW) \quad \frac{G|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta|\Gamma' \vdash \Delta'} \qquad (EC) \quad \frac{G|\Gamma \vdash \Delta|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta}$$

$$(S_c) \quad \frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2} \qquad (M) \quad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}$$

Logical rules

$$(\to, l) \quad \frac{G|\Gamma, B \vdash A, \Delta}{G|\Gamma, A \to B \vdash \Delta} \qquad (\to, r) \quad \frac{G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \to B, \Delta}$$

$$(+, l) \quad \frac{G|\Gamma, A, B \vdash \Delta}{G|\Gamma, A + B \vdash \Delta} \qquad (+, r) \quad \frac{G|\Gamma \vdash A, B, \Delta}{G|\Gamma \vdash A + B, \Delta}$$

$$(\wedge_i, l) \quad \frac{G|\Gamma, A_i \vdash \Delta}{G|\Gamma, A_1 \wedge A_2 \vdash \Delta} \qquad (\wedge, r) \quad \frac{G|\Gamma \vdash A, \Delta \quad G|\Gamma \vdash B, \Delta}{G|\Gamma \vdash A \wedge B, \Delta}$$

$$(\vee, l) \quad \frac{G|\Gamma, A \vdash \Delta \quad G|\Gamma, B \vdash \Delta}{G|\Gamma, A \vee B \vdash \Delta} \qquad (\vee_i, r) \quad \frac{G|\Gamma \vdash A_i, \Delta}{G|\Gamma \vdash A_1 \vee A_2 \vdash \Delta}$$

Theorem 2 ***LA*** *is sound and complete for* ***A*** *ie for a hypersequent* H, H *is provable in* ***LA*** *iff* ϕ^H *is valid in all abelian l-groups.*

References

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