Substitutions, test and knowledge in axiomatic products of *PDL* and *S5*

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Combinations of propositional dynamic logic (*PDL*) [3] and *S5* [1, 5] are meaningful when we want to reason about actions and knowledge. We focus on axiomatically defined products of *PDL* and *S5*, in which the *S5* modality commutes with the dynamic operators by the following axiom schema.

$$(\mathbf{NL} \wedge \mathbf{PR}) \qquad [a]\Box p \leftrightarrow \Box[a]p$$

(**NL** is short for no learning, and **PR** for perfect recall.) Generally axioms and theorems of a logic are assumed true for all instantiations for the atomic symbols. However, in this paper we distinguish between two variants of the substitution rule. The *weak substitution rule* allows the substitution of arbitrary formulae for the atomic propositional symbols but does not allow substitution for atomic action symbols. By contrast, the *full substitution rule* allows both kinds of substitutions.

Let $\mathcal{L}$ be the language of *PDL* plus an additional modal operator $\Box$ representing knowledge. A *logic* (*weak logic*) in $\mathcal{L}$ is a set of formulae which is closed under standard inference rules (modus ponens and necessity rule) and under the full (resp. weak) substitution rule. Notationally weak logics are discerned by a subscript w.

The simplest form of combination of two (or more) logics is their fusion, or independent join. We denote the *fusion* of *PDL* and *S5* by $PDL \oplus S5$.

Proposition 1. $PDL \oplus S5 = (PDL \oplus S5)_w$, *i.e. full substitution is admissible in* $(PDL \oplus S5)_w$.

$PDL \oplus S5$-models are combinations of the familiar Kripke models for *PDL* and *S5*, that is, a *$PDL \oplus S5$ model* is a tuple $\langle S, Q, R, \models\rangle$ such that $\langle S, Q, \models\rangle$ is a *PDL*-model [3] and $\langle S, R, \models\rangle$ is an *S5*-model [1].

The products of *PDL* and *S5* we consider are extensions of *(test-free)* $PDL \oplus S5$ with the axiom $(\mathbf{NL} \wedge \mathbf{PR})$ above. This axiom is a Sahlqvist formula and its first-order equivalent is:

$$(\mathbf{com}) \qquad R \circ Q(a) = Q(a) \circ R$$

In the case of logics with full substitution a ranges over all the actions, whereas for weak logics a ranges over the set of atomic actions only and in this case we will refer to this property as $\mathbf{com}_w$. We consider the logics:

$$[PDL, S5] \rightleftharpoons PDL \oplus S5 + \mathbf{NL} \wedge \mathbf{PR}$$
$$[\textit{test-free } PDL, S5] \rightleftharpoons \textit{test-free } PDL \oplus S5 + \mathbf{NL} \wedge \mathbf{PR}$$

and their weak versions $[PDL, S5]_w$ and $[\textit{test-free } PDL, S5]_w$. These extensions contain all possible definitions of axiomatic products of *(test-free)* *PDL* and *S5*. By definition, a *$[PDL, S5]$-model* (resp. *$[\textit{test-free } PDL, S5]$-model*) is a $PDL \oplus S5$-model (resp. test-free $PDL \oplus S5$-model) which satisfies property **com**. We have similar definitions for weak logics with the property $\mathbf{com}_w$ instead of **com**.

The following theorem tells us something about the admissibility of the full substituion rule in weak variants these logics.

Theorem 1. $[PDL, S5] \neq [PDL, S5]_w$, *but* $[\textit{test-free } PDL, S5] = [\textit{test-free } PDL, S5]_w$.

All considered logics can be proved complete, decidable, and have the small model property.

Theorem 2 (Small Model Property). *Let L be any of the logics $[PDL, S5]$, $[PDL, S5]_w$, or $[\textit{test-free } PDL, S5]$. Let ϕ be a formula of $\mathcal{L}$ and n the number of symbols in ϕ. If ϕ is satisfiable in some L-model then it is satisfiable in an L-model with no more than $\mu(L, n)$ states, where $\mu([PDL, S5], n) = 2^n$ and $\mu([PDL, S5]_w, n) = \mu([\textit{test-free } PDL, S5], n) = 2^n \cdot 2^{2^n}$.*

The knowledge modality can be trivially eliminated from $[PDL, S5]$.

Proposition 2. $p \leftrightarrow \Box p \in [PDL, S5]$.

This *trivial* elimination of the *S5* operator in the product $[PDL, S5]$ is unsatisfactory both from a logical perspective and an application-oriented perspective. The reason for the elimination of the

$S5$ operator is the implicit connection between the test operator and $\Box$ in the commutativity axioms under full substitutivity. A solution we propose here is to define an alternative semantics for test such that in the resulting logic, $\Box$ and test interact in a way so that weak substitutivity implies full substitutivity.

Therefore, we introduce a new operator, denoted by ⁇, as replacement for the standard test operator. The new operator can be interpreted as an *epistemic test* operator. The intuition of p⁇ is an action which can be succesfully accomplished only if p is *known* in the current state. The result of this action is an arbitary state within the same knowledge cluster. Thus, p⁇ is the action of confirming the agents's own knowledge. In contrast, with the usual test operator the agent has the capability to confirm truths rather than knowledge. Philosophically, this is a strong property of an agent; we believe, too strong. In agent based applications the new interpretation of test is more suitable than the usual test operator.

Subsequently, the logical apparatus is the same as previously with the obvious changes. The symbol ⁇ is used in the superscript to indicate a replacement of the operator ? by ⁇. Let $[PDL, S5]^{⁇}$ and $[PDL, S5]_w^{⁇}$ be the logics in $\mathcal{L}^{⁇}$ obtained from $[PDL, S5]$ and $[PDL, S5]_w$, respectively, by replacing the usual test axiom with:

$$[p⁇]q \leftrightarrow \Box(\Box p \rightarrow q).$$

In accordance with this axiom, the formula $[p⁇]q$ can be read as 'q is known with respect to p being known'. Thus, we think of the modal operator $[\_⁇]$ as the operator of *relative knowledge*.

Using the elimination of second-order quantifiers [2, 4] it is easy to find the corresponding semantic definition for the new operator. Thus, a $(PDL \oplus S5)^{⁇}$ model is a tuple $\langle S, Q, R, \models\rangle$ satisfying all the properties of $PDL \oplus S5$ model, except the meaning of ⁇ is specified by:

$$Q(\phi⁇) \rightleftharpoons \{(s, t) \in R \mid t \models \Box\phi\}.$$

This induces the notions of $[PDL, S5]_w^{⁇}$ and $[PDL, S5]^{⁇}$ models as expected.

The definition of ⁇ still allows the elimination of $\Box$ but this time the elimination is not trivial.

Proposition 3. $\Box p \leftrightarrow [\top⁇]p \in PDL \oplus S5^{⁇}$.

In this context, we obtain the theorem of admissibility of the full substitution rule in $[PDL, S5]_w^{⁇}$.

Theorem 3. $[PDL, S5]^{⁇} = [PDL, S5]_w^{⁇}$.

Also the following theorem is proved.

Theorem 4. *$[PDL, S5]^{⁇}$ has small model property with $\mu([PDL, S5]^{⁇}, n) = 2^n \cdot 2^{2^n}$ and is complete and decidable.*

How does the standard test operator of PDL relate to the new one? It turns out that there is a simulation of PDL in $(PDL \oplus S5)^{⁇}$. Define the translation σ from formulae of PDL to $\mathrm{For}^{⁇}$ by the following:

$$\begin{array}{rcl} \sigma a = a & & \sigma\bot = \bot \\ \sigma(\psi?) = (\sigma\psi)⁇ & & \sigma p = \Box p \\ \sigma(\alpha \cup \beta) = \sigma\alpha \cup \sigma\beta & & \sigma(\phi \rightarrow \psi) = \Box(\sigma\phi \rightarrow \sigma\psi) \\ \sigma(\alpha;\beta) = \sigma\alpha;\sigma\beta & & \sigma([\alpha]\psi) = \Box[\sigma\alpha]\sigma\psi \\ \sigma(\alpha^*) = (\sigma\alpha)^* & & \end{array}$$

Theorem 5. *For any $\Box$-free formula ϕ in $\mathcal{L}$, $\phi \in PDL$ iff $\sigma\phi \in (PDL \oplus S5)^{⁇}$.*

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