Proof Planning Diagonalisation Theorems via Category Theory

Alan Smaill   A.Smaill@ed.ac.uk
Division of Informatics, University of Edinburgh

1 "The" diagonal argument?

Various automated reasoning approaches to diagonalisation arguments have approached the subject via a meta-theoretic account of proof state, or reformulation via syntactic manipulation (*eg* [HKC95, CS98, Mel94]). An alternative approach is based upon a more abstract characterisation of diagonal arguments, where a single mathematical argument at the level of category theory can be instantiated into a wide range of diagonal arguments. Category theory shares with proof planning the desire to generalise patterns of argument, though it goes about this in a different way. The claim here is that this different separation between object-level mathematics and meta-level control gives us more mathematical feel for the shape of the proofs concerned.

Lawvere and Schanuel's book [LS97], which we follow here, is intended as a way into mathematics for "the general reader or beginning student". In it (in section 29) there is a good discussion of diagonalisation, based on Lawvere's earlier research paper [Law69]. I assume basics of category theory in the following summary. Suppose we are in a category with products, *ie* we have:

- Objects $X, Y, T, \ldots$; with an associated notion of
- Maps $f, g \ldots$, between these objects which can be composed in sensible ways (including identity maps from an object to itself); and
- Product objects like $X \times Y$. These are characterised via projection maps, *eg* $\pi_X : X \times Y \to X$ and the ability to form a map $\langle f, g\rangle : A \to X \times Y$ given maps $f : A \to X$, $g : A \to Y$.

Now the diagonal map $\delta : X \to X$ is just $\langle 1_X, 1_X\rangle$ where 1_X is the identity map on X. We suppose there is an object $\mathbf{1}$ with exactly one map $X \to \mathbf{1}$ for every X. For any $\phi : T \times X \to Y$, there is a family of maps $T \to Y$ indexed by maps $t : \mathbf{1} \to X$. If *all* the maps $T \to Y$ are in this family, we say that $\phi : T \times X \to Y$ *parametrises* the maps $T \to Y$.

From this can be shown both positive and negative forms of the diagonal argument. The latter says that:

> **Diagonal Argument, Contrapositive.** In a category C with products, if there is some $\alpha : Y \to Y$ with no fixed points, then for every T and every purported parametrisation $\phi : T \times T \to Y$, there is a map $f : T \to Y$ which is left out of the parametrised family.

Out of this drops for example that looking at sets and functions, no map $T \times T \to$ *bool* (where *bool* is simply a two-element set) can parametrise all maps $T \to$ *bool*, since the swap map on *bool* has no fixed-point.

2 Proof planning around this argument

It is plausible that this version of the diagonal argument is sufficiently general that reformulation is not needed, but rather we want to instantiate the ingredients of the argument in a particular context. Typically we need to, in some order:

1. Identify the objects and maps involved.
2. Check that maps compose, that identity map is present.
3. Identify the appropriate product construction, and check its characteristic properties.
4. Look for a fixed-point free map from the appropriate object to itself.

If these all succeed, we have the components for a diagonal argument, in the negative form. If (4) fails, the negative argument fails (with the given choice of objects and maps), though the conclusion may hold anyway.

It is natural to use a higher-order representation in dealing with this sort of problem. An initial implementation in λ*Clam* shows that this approach is feasible. Notably, it allows us to invent examples (and non-examples) of diagonal arguments outside the standard stock. For example, can we continuously parametrise the set of continuous functions $f : \mathcal{R} \to \mathcal{R}$ by means of some $\phi : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$? We are pointed in the direction of a series of lemmas:

1. Composition of continuous functions are continuous, and the identity map is continuous.
2. The topology on $X \times Y$ is defined so that the usual projections are continuous, and do indeed characterise the product correctly.
3. There are continuous maps $\mathcal{R} \to \mathcal{R}$ with no fixed-points (eg $x \mapsto x + 1$).

And so we conclude that there can be no such continuous ϕ.

[LS97] does a sketch of how incompleteness for arithmetic fits into this set-up. It would be much harder to invent the appropriate constructions here, but not so hard to check that they do fit this pattern.

In conclusion, this different way of organising a family of proofs by seeing them as instances of a single more abstract argument is amenable to implementation in a higher-order framework. It allows us to apply diagonal arguments across a wide range of examples, and gives an explanation for some cases where diagonalisation fails.

References

[CS98] Lassaad Cheikhrouhou and Jörg Siekmann. Planning diagonalization proofs. In *Proceedings of 8th International Conference on Artificial Intelligence: Methodology, Systems, Applications (AIMSA'98)*, volume 1480 of *Lecture Notes in Artificial Intelligence*, pages 167–180. Springer, 1998.

[HKC95] X. Huang, M. Kerber, and L. Cheikhrouhou. Adapting the diagonalization method by reformulations. In A. Levy and P. Nayak, editors, *Proc. of the Symposium on Abstraction, Reformulation and Approximation (SARA-95)*, pages 78–85. Ville d'Esterel, Canada, 1995.

[Law69] F.W. Lawvere. Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications II*, volume 92 of *Lecture Notes in Mathematics*, pages 143–145. Springer, 1969.

[LS97] F.W. Lawvere and S.H. Schanuel. *Conceptual Mathematics*. Cambridge University Press, Cambridge, England, 1997.

[Mel94] E. Melis. Representing and transferring diagonal methods. Technical Report CMU-CS-94-174, Carnegie Mellon University, 1994.