Embedding a quantified logic for spatial reasoning in Isabelle-HOL

Paolo Torrini, Jacques D. Fleuriot,
John G. Stell, Brandon Bennett.
{jgs,brandon,paolot}@comp.leeds.ac.uk (Sc. of Computing, Un. of Leeds)
{jdf,paolot}@dai.ed.ac.uk (Sc. of AI, Un. of Edinburgh)

Spatial semantics have often been discussed, quite recently, both from the perspective of first-order systems [RCC92, PS98] and from that of non-classical propositional logics [Ben96, Lem96, She99, Ste00]. One of the general directions seems to be toward the development of models of space that are closer to common-sense reasoning than the standard one based on a cartesian product of the reals. Qualitative models can either refer to continuous spaces or to digital ones; the latter ones, on which we intend to focus here, can also be intended as multi-resolution cell complexes, and are in general comparatively closer to machine representations [Kov92]. The basic semantical entities are taken to be *regions* that can be topologically interpreted as regular open sets. Some of the spatial relations that can be defined on regions are: parthood, disjointness, disconnection (binary), emptiness, connectedness (unary), and their complements. Topologically, parthood can be interpreted as inclusion, disjointness as empty intersection, disconnection as empty intersection of the respective closures, connectedness as impossibility to be split into two non-empty, disjoint open sets.

It has already been discussed [TB02, TS02], in first place, how the topological semantics for intuitionistic propositional logic (introduced with [Tar56]) can be re-adapted into a semantics for an axiomatic version of intuitionistic second-order propositional logic (ISPL); besides, how an extension of ISPL can be used to encode all the above-mentioned 'positive' relations (and, adding some semantical constraint, also the complementary ones); the encoding does not work for all the spaces, but it does for a subclass of Alexandroff topologies which are close enough to realise an intuitive idea of digital spaces.

Provability in ISPL can be inductively defined in terms of natural deduction, sequent-calculus or Hilbert axiomatisation [Loe76, Gab81]; the definition of formula is of course different from that of intuitionistic first-order predicate calculus (IFPC), since variables have type formula (wff) instead than type term (term), but the postulates are basically the same; the interpretation of quantifiers is substitutional; actually, not all the IFPC postulates are necessary, since all the logical operators are 2nd-order definable from $\rightarrow, \forall$; however, we follow [Gab81] in adding as an extra postulate the following 'comprehension' principle $\vdash \exists x.(x \leftrightarrow A)$

Isabelle is an interactive, generic theorem-prover [Pau94] that has an implementation of higher-order logic (HOL), allowing higher-order syntax for the binders, and making it possible to deal semi-automatically with positive inductive definitions. An embedding of ISPL in Isabelle-HOL can be interesting especially insofar as it allows the mechanisation of inductive meta-proofs; an example is the proof required by the *replacement of equivalents* ($\vdash A \leftrightarrow B$ implies $\vdash C(A/x) \leftrightarrow C(B/x)$), a principle that can be used to simplify quite radically the object-level reasoning.

Nevertheless, there seems to be a significant problem with handling the embedding of ISPL in a direct way; in terms of higher-order syntax, propositional quantification has an 'impredicative' type $(\mathsf{wff} \mapsto \mathsf{wff}) \mapsto \mathsf{wff}$, which is not admissible from the point of view of recursive type declarations for positive inductive definitions. A way to get around this problem appears to be a translation of ISPL into IFPC; the latter can in fact be embedded in Isabelle-HOL in a comparatively standard way, using the rules of sequent calculus. One possibility seems that to rely on a representation with all the rules for IFPC, a one-place predicate *Var* (where $Var(x)$ is intended to be the propositional variable for which the object-variable x stands), and a corresponding version of the comprehension axiom. Another idea is that of using the IFPC rules for $\rightarrow$ and $\forall$, with equality, $\iota : (\mathsf{term} \mapsto \mathsf{wff}) \mapsto \mathsf{term}$ (the description operator, which can be axiomatised with $\vdash x = \iota y.P(y)$ iff $\vdash P(x) \wedge (\forall y.P(y) \rightarrow y = x))$, *Var*, and a one-place, second-order propositional function which associates a formula A to the object-variable x such that $Var(x) = A$, defined as $Term := \lambda A.\iota x.(Var(x) \leftrightarrow A)$, assuming additionally that ι is defined for every instance of $\lambda x.(Var(x) \leftrightarrow A)$ (with x not free in A), so that $A \leftrightarrow Var(Term A)$ is provable for every A. At the moment of writing, a prototype has been implemented, closer to the second line of thought (though introducing *Term* directly, without ι, and overloading the built-in notion of substitution with a recursive definition that does not affect the treatment of the binders); the automatically generated induction principle provided by Isabelle has been successfully tested on the proof of the replacement of equivalents.

References

[Ben96] B. Bennett. Modal logics for qualitative spatial reasoning. *Bulletin of the Interest Group in Pure and Applied Logic (IGPL)*, 4(1):23–45, 1996.

[Gab81] D.M. Gabbay. *Semantical Investigation in Heyting intuitionistic logic*. Number 148 in Synthese. Reidel, 1981.

[Kov92] V.A. Kovalevsky. Topological foundations of shape analysis. In O. Ying-Lie, A. Toet, D. Foster, H.J.A.M. Heijmans, and P. Meer, editors, *Shape in picture*, pages 21–37. Springer-Verlag, 1992.

[Lem96] O. Lemon. Semantical foundations of spatial logic. In L. Aiello, J. Doyle, and S. Shapiro, editors, *Principles of Knowledge Representation and Reasoning: Proceedings of the 6th International Conference (KR-96)*, pages 212–219. Morgan Kaufman, 1996.

[Loe76] M. H. Loeb. Embedding first order predicate logic in fragment of intuitionistic logic. *Journal of Symbolic Logic*, 41:705–718, 1976.

[Pau94] L.C. Paulson. *Isabelle: a generic theorem prover*, volume 828. Springer, 1994.

[PS98] I. Pratt and D. Schoop. A complete axiom system for polygonal mereotopology of the real plane. *Journal of Philosophical Logic*, 27:621–658, 1998.

[RCC92] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In *Proc. 3rd Int. Conf. on Knowledge Representation and Reasoning*, pages 165–176. Morgan Kaufmann, 1992.

[She99] V. Shehtman. 'Everywhere' and 'here'. *Journal of Applied Non-Classical Logics*, 9(2-3):369–279, 1999.

[Ste00] J. G. Stell. Boolean connection algebras: a new approach to the region-connection calculus. *Artificial Intelligence*, (122):111–136, 2000.

[Tar56] A. Tarski. Sentential calculus and topology. In *Logic, Semantics, Metamathematics*. Oxford University Press, 1956.

[TB02] P. Torrini and B. Bennett. A spatial logic capable of representing connectedness and planarity. Submitted for publication, 2002.

[TS02] P. Torrini and J.G. Stell. Spatial concepts in an intermediate logic with propositional quantification. Submitted for publication, 2002.