A Principle for Incorporating Axioms into the First-Order Translation of Modal Formulae

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Introduction. Modal logic is widely accepted to provide an appropriate formal framework for an ever increasing number of different application areas of computer science. In the field of knowledge representation, particularly in the subfield concerned with description logics, modal logics arise in the form of the basic multi-modal logic and propositional dynamic logic. Modal logics also form the basis of many agent logics. Various aspects of agents are formalised by modal operators, e.g. *S5* modalities for knowledge, *KD45* modalities for belief, and so on. Normally agent logics also include operators formalising notions such as ability and commitment which are given by non-standard modal operators. Agent-based systems are a particular application area, where there is a clear need for non-standard modal logics, that is, modal logics that do not just simply coincide with familiar modal logics. There is therefore an increasing interest in non-standard combinations of modal logics. These modal logics are commonly given through a set of modal axiom schemata in the form of a Hilbert axiomatisation. However, it is generally accepted that Hilbert axiomatisations are not amenable for automated reasoning. Anyone who has studied Hilbert-style axiomatisations of modal logics or other logics knows that, except for very simple theorems, if is difficult to derive theorems in this framework. With lots of practice one can become quite good at doing Hilbert-style proofs, but in general, writing down proofs using this kind of approach requires a lot of ingenuity. This is also why Hilbert-style deduction is difficult to automate effectively. While they are easy to automate by reduction to first-order logic, the search space is enormous. Consequently, there are no effective implementations of Hilbert systems. The question arises how effective reasoning systems can be constructed for a given combination of (possibly interacting) modal logics, specified by a Hilbert axiomatisation. The most common approach is to adopt a semantics-based method. For this it is necessary to know what the Kripke models for the given Hilbert axiomatisation are. The popular tableau approaches (and many sequent-style approaches) can be seen as methods for constructing a Kripke model for a given formula. Another option is to use a semantics-based translation into first-order or second-order logic and then to use first-order or second-order theorem provers. In all these cases it is necessary to find a class of Kripke models with respect to which the given Hilbert axiomatisation is sound and complete. Again one would want to automate this process, or at least have some sort of automated support. A number of automated second-order quantifier elimination procedures were developed and studied that transform Hilbert axioms into corresponding properties of the accessibility relation in the Kripke semantics of modal logics.

We propose an alternative idea that is based on an approach in which, starting from a Hilbert axiomatisation, by a process of partial translation and proof-theoretic transformation we can directly obtain proof methods for a wide variety of modal logics. We introduce and study a principle, called the *axiomatic translation principle*, of reducing modal problems into first-order logic. What we try to achieve with the axiomatic translation principle is to find a simplified theory, which captures the semantic background theory of a given modal logic in a new way, in particular, in a way that is more amenable to automation. The axiomatic translation principle is conceptually very simple; it is based on the standard relational translation method, but instead of using correspondence properties, the principle incorporates modal axioms into the first-order translation of modal formulae. This provides a method for avoiding having to handle properties such as transitivity, Euclideanness and functionality, which are difficult to handle efficiently by first-order theorem provers. It is not known how to use first-order theorem provers as decision procedures for these properties. For example, no resolution decision procedures are known for decidable first-order fragments relevant to modal logics which include the formula $\forall xyz\,(R(x,y) \wedge R(y,z) \rightarrow R(x,z))$ expressing the transitivity of a relation R. The clauses in the clausal form of this formula, and also Euclideanness and functionality, contains no single maximal literal under common orderings. This means these clauses are eligible for many more inferences compared to clauses which contain one maximal literal. Examples of the latter kind are properties such as reflexivity, symmetry and seriality.

The axiomatic translation principle overcomes the difficulties of using correspondence properties. Modal axiom schemata are incorporated into the translation of a modal logic problem by partial instantiations of the modal axiom schema, which are given by partial instantiations of characteristic so-called schema clauses. These schema clauses can be automatically deduced from the modal axioms, and we show that soundness of the presented method is automatic under some weak, reasonable assumptions. More challenging is proving completeness. The reward for succeeding in proving completeness is automatic decidability, and other nice properties such as interpolation and sound and complete tableau systems.

Results.

Theorem 1 (Soundness and Completeness). *Let L be any of the modal logics $K4$, KT, KD, KB, $Kalt_1$, $K4^{\kappa}$, $Kalt_1^{\kappa_1,\kappa_2}$, $KT4$, KTB, KDB, $KD4$, $K5$, $K5^{\kappa}$, $K4B$, $KT4B$. For any modal formula φ, there is an axiomatic translation mapping Π_L such that (i) φ is satis-*

fiable in L iff $\Pi_L(\varphi)$ is first-order satisfiable, (ii) $\Pi_L(\varphi)$ can be computed in linear time.

The axiomatic translation reduces modal problems into decidable fragments of first-order logic (DL^* or GF^2). A general decidability result can therefore be formulated:

Theorem 2 (Decidability). *Any sound and complete propositional modal logic is decidable when an effectively computable, complete axiomatic translation mapping exists.*

Corollary 1. *The modal logics of Theorem 1 and their fusions are decidable.*

Theorem 3 (Small model property). *The modal logics of Theorem 1 and their fusions have the small model property, more precisely, if a modal formula φ is L-satisfiable then it is satisfiable in a model with at most $2^{|\varphi|}$ elements.*

Theorem 4 (Local interpolation). *The modal logics of Theorem 1 and their fusions have local interpolation.*

Decidability of extensions of K with the axioms T, B, D, 4, 5, alt_1 are well-known. In Gabbay [2] the decidability of the logics $K4^\kappa$, $K5^\kappa$ and $K\,alt_1^{\kappa_1,\kappa_2}$ was shown by using a reduction to SωS. It is also well-known that the operation of fusing normal propositional modal logics preserves decidability. Thus the decidability of the logics considered is known or follows from known results. As for the other results, we believe the small model property and local interpolation of the logics $K4^\kappa$, $K5^\kappa$ and $K\,alt_1^{\kappa_1,\kappa_2}$ have been open up to now.

Given a finite axiomatisation and a small model property result a decision procedure can be defined, but not a very useful procedure. However since the axiomatic translation embeds the modal problems into DL^* or GF^2 and since both can be decided with resolution the following is immediate.

Theorem 5 (Decidability by resolution). *Resolution based on any refinement compatible with the ordering $>_d$ defined in [1] decides the axiomatic encoding of satisfiability problems in all modal logics satisfying the conditions of Theorem 2.*

Most modern first-order theorem provers implement ordered resolution and because the ordering $>_d$ is standardly available in theorem provers, these provers immediately provide practical decision procedures for solving the satisfiability problem of many modal logics. All that needs implementing is the axiomatic translation of modal problems into first-order formulae, which is not difficult. It should be noted that the result is not true for hyperresolution which is used in the completeness proofs. However with the blocking inference rule introduced in [4] hyperresolution can be turned into a decision procedure for the axiomatic translation of the considered logics.

Practical experience. The attractiveness of the axiomatic translation principle is its amenability for automation. We have developed a prototype implementation in Prolog of the axiomatic translation. As a theorem prover any existing first-order theorem prover can be used. We used (M)SPASS and VAMPIRE, which are among the best first-order resolution theorem provers, in our tests. Both (M)SPASS and VAMPIRE support ordered resolution refinements compatible with the ordering $>_d$, and in fair mode, both provide a decision procedure for the axiomatic translation of all modal logics which satisfy the general conditions of Theorem 5.

Initial experiments show that performance improvements over the classical semantics-based translation approach are considerable. For many problems improvements by factors of 10–1000 have been obtained with both provers, and some problems which are not solvable with the classical translation approach become solvable with the new axiomatic translation approach.

Concluding remarks. The results obtained are strong indicators of the promise that the axiomatic translation principle holds. The results cover a large class of propositional modal logics which satisfy some general conditions. The investigation of the axiomatic translation on the level of first-order logic provides the opportunity for mixing different translation mappings. For example, we have shown that it is possible to mix the axiomatic translation with the classical translation approach and, exploiting the nice termination properties of ordered resolution for expressive first-order fragments, it is possible to state a general decidability result for the mixed approach. Sound and complete tableau calculi can be easily extracted from the combination of the axiomatic translation and hyperresolution by following the approach of [1]. The axiomatic translation principle can also be used to combine different styles of reasoning systems. For example, in [3] the axiomatic translation was used in a decision procedure for the join of discrete linear temporal logic with subsystems of *S5*. The system combines temporal resolution for the temporal part and first-order resolution on the modal part.

More details about the axiomatic translation principle can be found in [5, 6].

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